THE STRUCTURE OF GELFAND-LEVITAN-MARCHENKO TYPE EQUATIONS FOR DELSARTE TRANSMUTATION OPERATORS OF LINEAR MULTI-DIMENSIONAL DIFFERENTIAL OPERATORS AND OPERATOR PENCILS. PART 1.

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Abstract. An analog of Gelfand-Levitan-Marchenko integral equations for multi- dimensional Delsarte transmutation operators is constructed by means of studying their differential-geometric structure based on the classical Lagrange identity for a formally conjugated pair of differential operators. An extension of the method for the case of affine pencils of differential operators is suggested.

1. Introduction

Consider the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m; \mathbb{C}^N)$, $m, N \in \mathbb{Z}_+$, and the correspondingly conjugated pair $\mathcal{H}^* \times \mathcal{H}$ on which one can define the natural scalar product

$$(\varphi, \psi) = \int_{\mathbb{R}^m} dx < \varphi, \psi > := \int_{\mathbb{R}^m} dx \bar{\varphi}^{\mathsf{T}}(x)\psi(x), \tag{1.1}$$

where $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, the sign " $-$ " means the complex conjugation and the sign " T" means the standard matrix transposition. Take also two linear densely defined differential operators L and $\tilde{L} : \mathcal{H} \to \mathcal{H}$ and some two closed functional subspaces $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0 \subset \mathcal{H}_-$, where $\mathcal{H}_-$ is the negative Hilbert space from a Gelfand triple

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \tag{1.2}$$

of the correspondingly Hilbert-Schmidt rigged [19, 1] Hilbert space $\mathcal{H}$. We will use further the following definition.

Definition 1.1. (see J. Delsarte and J. Lions [2, 3]). A linear invertible operator $\mathbf{\Omega}$, defined on $\mathcal{H}$ and acting from $\mathcal{H}_0$ onto $\tilde{\mathcal{H}}_0$, is called a Delsarte transmutation operator for a pair of linear differential operators L and $\tilde{L} : \mathcal{H} \longrightarrow \mathcal{H}$, if the following two conditions hold:

- the operator $\mathbf{\Omega}$ and its inverse $\mathbf{\Omega}^{-1}$ are continuous in $\mathcal{H}$;
- the operator identity

$$\tilde{L}\mathbf{\Omega} = \mathbf{\Omega} L \tag{1.3}$$

is satisfied.

1991 *Mathematics Subject Classification.* Primary 34A30, 34B05 Secondary 34B15
PACS: 02.30.Jr, 02.30.Uu, 02.30.Zz, 02.40.Sf.
Key words and phrases. Delsarte transmutation operators, parametric functional spaces, Darboux transformations, inverse spectral transform problem, soliton equations, Zakharov-Shabat equations.

Such transmutation operators were for the first time introduced in [2, 3] for the case of one-dimensional differential operators. In particular, for the Sturm-Liouville and Dirac operators the complete structure of the corresponding Delsarte transmutation operators was described in [5, 4], where also the extensive applications to spectral theory were done. As was shown in [5, 6, 4], for the case of one-dimensional differential operators an important role in theory of Delsarte transmutation operators is played by special integral Gelfand-Levitan-Marchenko (GLM) equations [17, 4, 5], whose solutions are exactly kernels of the corresponding Delsarte transmutation operators. Some results for two-dimensional Dirac and Laplace type operators, were also obtained in [15, 6] .

In the present work, based on results of [10, 8, 11, 9], we shall construct for a pair of multi-dimensional differential operators acting in a Hilbert space $\mathcal{H}$ a special pair of conjugated Delsarte transmutation operators $\mathbf{\Omega}_+$ and $\mathbf{\Omega}_-$ in $\mathcal{H}$ and a pair $\mathbf{\Omega}_+^{\circledast}$ and $\mathbf{\Omega}_-^{\circledast}$ in $\mathcal{H}^*$ parametrized by two pairs of closed subspaces $\mathcal{H}_0$, $\tilde{\mathcal{H}}_0 \subset \mathcal{H}_-$ and $\mathcal{H}_0^*$, $\tilde{\mathcal{H}}_0^* \subset \mathcal{H}_-^*$, such that the operators $\mathbf{\Phi} := \mathbf{\Omega}_+^{-1}\mathbf{\Omega}_- - \mathbf{1}$ from $\mathcal{H}$ to $\mathcal{H}$ and $\mathbf{\Phi}^{\circledast} := \mathbf{\Omega}_+^{\circledast,-1}\mathbf{\Omega}_-^{\circledast} - \mathbf{1}$ from $\mathcal{H}^*$ to $\mathcal{H}^*$ are ones of Hilbert-Shmidt type, thereby determining via the equalities

$$\mathbf{\Omega}_+(1+\mathbf{\Phi}) = \mathbf{\Omega}_-, \quad \mathbf{\Omega}_+^{\circledast}(1+\mathbf{\Phi}^{\circledast}) = \mathbf{\Omega}_-^{\circledast} \tag{1.4}$$

the corresponding analogs of GLM-equations, taking into account that supports of both kernels of integral operators $\mathbf{\Omega}_+, \mathbf{\Omega}_-$ and $\mathbf{\Omega}_+^{\circledast}$, $\mathbf{\Omega}_-^{\circledast}$ are correspondingly disjoint. Moreover, the following important expressions

$$\begin{aligned} \mathbf{\Omega}_+ L \mathbf{\Omega}_+^{-1} &= \tilde{L} = \mathbf{\Omega}_- L \mathbf{\Omega}_-^{-1}, \\ (\mathbf{1}+\mathbf{\Phi})L &= L(\mathbf{1}+\mathbf{\Phi}), \quad (\mathbf{1}+\mathbf{\Phi}^{\circledast})L^* = L^*(\mathbf{1}+\mathbf{\Phi}^{\circledast}) \end{aligned} \tag{1.5}$$

hold. As in the classical case [4, 5, 17], the solutions to this GLM-equation also give rise to kernels of the corresponding Delsarte transmutation operators $\mathbf{\Omega}_\pm$ in $\mathcal{H}$, that are very important [1, 18] for diverse applications.

Another trend of this work is related with a similar problem of constructing Delsarte transmutation operators and corresponding integral GLM-equations for affine pencils of linear multi-dimensional differential operators in $\mathcal{H}$, having important applications, in particular, for the inverse spectral problem and feedback control theory [7].

2. Generalized Lagrangian identity, its differential-geometric structure and Delsarte transmutation operators

Consider a multi-dimensional differential operator $L : \mathcal{H} \longrightarrow \mathcal{H}$ of order $n(L) \in \mathbb{Z}_+$:

$$L(x;\partial) := \sum_{|\alpha|=0}^{n(L)} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{2.1}$$

defined on a dense subspace $D(L) \subset \mathcal{H}$, where, as usually, one assumes that coefficients $a_\alpha \in \mathcal{S}(\mathbb{R}^m; End\mathbb{C}^N)$, $\alpha \in \mathbb{Z}_+^m$ is a multi-index, $|\alpha| = \overline{0, n(L)}$, and $x \in \mathbb{R}^m$. The formally conjugated to (2.1) operator $L^* : \mathcal{H}^* \longrightarrow \mathcal{H}^*$ is of the form

$$L^*(x;\partial) := \sum_{|\alpha|=0}^{n(L)} (-1)^{|\alpha|} (\frac{\partial^{|\alpha|}}{\partial x^\alpha} \cdot \bar{a}_\alpha(x)), \tag{2.2}$$

$x \in \mathbb{R}^m$ and the dot " $\cdot$ " above means the usual composition of operators.

Subject to the standard semilinear form $< \cdot, \cdot >$ on $\mathbb{C}^N \times \mathbb{C}^N$ one can write down easily the following generalized Lagrangian identity:

$$< L^*\varphi, \psi > - < \varphi, L\psi > = \sum_{i=1}^{m} (-1)^{i+1} \frac{\partial}{\partial x_i} Z_i[\varphi, \psi], \tag{2.3}$$

where for any pair $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$ expressions $Z_i[\varphi, \psi]$, $i = \overline{1, m}$, being semi-linear on $\mathcal{H}^* \times \mathcal{H}$. Having now multiplied (2.3) by the oriented Lebesgue measure $dx := \underset{j=\overrightarrow{1,m}}{\wedge} dx_j$, we easily get that

$$[< L^*\varphi, \psi > - < \varphi, L\psi >]dx = dZ^{(m-1)}[\varphi, \psi], \tag{2.4}$$

where

$$Z^{(m-1)}[\varphi, \psi] := \sum_{i=1}^{m} dx_1 \wedge dx_2 \wedge ... \wedge dx_{i-1} \wedge Z_i[\varphi, \psi] dx_{i+1} \wedge ... \wedge dx_m \tag{2.5}$$

is a $(m-1)$-differential form [12, 13] on $\mathbb{R}^m$ with meanings in $\mathbb{C}$.

Assume now that a pair $(\varphi, \psi) \in \mathcal{H}_0^* \times \mathcal{H}_0 \subset \mathcal{H}_-^* \times \mathcal{H}_-$, where, by definition,

$$\begin{aligned} \mathcal{H}_0 &:= \{\psi(\xi) \in \mathcal{H}_- : L\psi(\xi) = 0, \quad \psi(\xi)|_\Gamma = 0, \ \xi \in \Sigma \subset \mathbb{C}^p\}, \\ \mathcal{H}_0^* &:= \{\varphi(\eta) \in \mathcal{H}_-^* : L^*\varphi(\eta) = 0, \quad \varphi(\eta)|_\Gamma = 0, \ \eta \in \Sigma \subset \mathbb{C}^p\} \end{aligned} \tag{2.6}$$

with $\Sigma \subset \mathbb{C}^p$ being some "spectral" parameter space, $\Gamma \subset \mathbb{R}^m$ being some $(m-1)$-dimensional hypersurface piece-wise smoothly imbedded into $\mathbb{R}^m$, and $\mathcal{H}_-^* \supset \mathcal{H}^*$, $\mathcal{H}_- \supset \mathcal{H}$, being as before the correspondingly Hilbert-Schmidt rigged [1, 19, 17] Hilbert spaces, containing so called generalized eigenfunctions of the operators L^* and L. Thereby, for any pair $(\varphi, \psi) \in \mathcal{H}_0^* \times \mathcal{H}_0$ one gets from (2.4) that the differential $m-1$-form $Z^{(m-1)}[\varphi, \psi]$ is closed in the Grassmann algebra $\Lambda(\mathbb{R}^m; \mathbb{C})$. As a result from the Poincare lemma [12, 13], one finds that there exists an $(m-2)$-differential form $\Omega^{(m-2)}[\varphi, \psi] \in \Lambda^{m-2}(\mathbb{R}^m; \mathbb{C})$, semi-linearly depending on $\mathcal{H}_0^* \times \mathcal{H}_0$, such that

$$Z^{(m-1)}[\varphi, \psi] = d\Omega^{(m-2)}[\varphi, \psi]. \tag{2.7}$$

Making use now of the expression (2.7), we can get due to the Stokes theorem [12, 13], that

$$\begin{aligned} &\int_{\mathcal{S}_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\eta), \psi(\xi)] = \\ &\int_{\sigma_x^{(m-2)}} \Omega^{(m-2)}[\varphi(\eta), \psi(\xi)] - \int_{\sigma_{x_0}^{(m-2)}} \Omega^{(m-2)}[\varphi(\eta), \psi(\xi)] := \\ &\Omega_x(\eta, \xi) - \Omega_{x_0}(\eta, \xi), \end{aligned} \tag{2.8}$$

for all $(\eta, \xi) \in \Sigma \times \Sigma$, where an $(n-1)$-dimensional hypersurface $\mathcal{S}_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \subset \mathbb{R}^m$ with the boundary $\partial \mathcal{S}_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) = \sigma_x^{(m-2)} - \sigma_{x_0}^{(m-2)}$ is defined as a film spanned in some way between two $(m-2)$-dimensional homological nonintersecting each other cycles $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \subset \mathbb{R}^m$, parametrized, correspondingly, by some arbitrary but fixed points x and $x_0 \in \mathbb{R}^m$. The quantities $\Omega_x(\eta, \xi)$ and $\Omega_{x_0}(\eta, \xi)$, $(\eta, \xi) \in \Sigma \times \Sigma$, obtained above have to be considered naturally as the corresponding kernels [1, 16, 17] of bounded Hilbert-Schmidt type integral operators $\Omega_x, \Omega_{x_0} : H \to H$, where $H := L_2^{(\rho)}(\Sigma; \mathbb{C})$ is a Hilbert space of functions on

Σ measurable with respect to a finite Borel measure ρ on Borel subsets of Σ, and satisfying the following weak regularity condition

$$lim_{x\to x_0}\Omega(\eta,\xi)] = \Omega_{x_0}(\eta,\xi) \tag{2.9}$$

for any pair $(\varphi(\eta),\psi(\xi)) \in \mathcal{H}_0^*\times\mathcal{H}_0$, $(\eta,\xi)\in\Sigma\times\Sigma$.

Now we are, similarly to results of [8, 10, 11], in a position to construct the corresponding pair of spaces $\tilde{\mathcal{H}}_0^*$ and $\tilde{\mathcal{H}}_0\subset\mathcal{H}$, related with a Delsarte transformed linear differential operator $\tilde{L}:\mathcal{H}\longrightarrow\mathcal{H}$ and its conjugated expression $\tilde{L}^*:\mathcal{H}^*\longrightarrow\mathcal{H}^*$,

$$\tilde{L}(x;\partial) := \sum_{|\alpha|=0}^{n(\tilde{L})}\tilde{a}_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{2.10}$$

with coefficients $\tilde{a}_\alpha\in\mathcal{S}(\mathbb{R}^m;End\mathbb{C}^N)$, $\alpha\in\mathbb{Z}_+^m$ is a multi-index, $|\alpha| = \overline{0,n(\tilde{L})}$, $x\in\mathbb{R}^m$, under the condition $n(\tilde{L}) = n(L)\in\mathbb{Z}_+$ be fixed. Namely, let closed subspaces $\tilde{\mathcal{H}}_0^*\subset\tilde{\mathcal{H}}_-^*$ and $\tilde{\mathcal{H}}_0\subset\tilde{\mathcal{H}}_-$ be defined as

$$\begin{gathered}\tilde{\mathcal{H}}_0 := \{\tilde{\psi}(\xi)\in\mathcal{H}_- : \tilde{\psi}(\xi) = \psi(\xi)\cdot\Omega_x^{-1}\Omega_{x_0},\\ (\varphi(\eta),\psi(\xi))\in\mathcal{H}_0^*\times\mathcal{H}_0,\ (\eta,\xi)\in\Sigma\times\Sigma\},\\ \tilde{\mathcal{H}}_0^* := \{\tilde{\varphi}(\eta)\in\mathcal{H}_-^* : \tilde{\varphi}(\eta) = \varphi(\eta)\cdot\Omega_x^{\circledast,-1}\Omega_{x_0}^{\circledast},\\ (\varphi(\eta),\psi(\xi))\in\mathcal{H}_0^*\times\mathcal{H}_0,\ (\eta,\xi)\in\Sigma\times\Sigma\}\}.\end{gathered} \tag{2.11}$$

Here, similarly to (2.8), we defined the kernels of bounded invertible integral operators $\Omega_x^{\circledast}$ and $\Omega_{x_0}^{\circledast}:H\longrightarrow H$ as follows:

$$\begin{aligned}&\int_{\mathcal{S}_+(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\bar{Z}^{(m-1),\intercal}[\varphi(\eta),\psi(\xi)]\\ &= \int_{\sigma_x^{(m-2)}}\bar{\Omega}^{(m-2),\intercal}[\varphi(\eta),\psi(\xi)] - \int_{\sigma_{x_0}^{(m-2)}}\bar{\Omega}^{(m-2),\intercal}[\varphi(\eta),\psi(\xi)]\\ &:= \Omega_x^{\circledast}(\eta,\xi) - \Omega_{x_0}^{\circledast}(\eta,\xi)\end{aligned}$$

for all $(\eta,\xi)\in\Sigma\times\Sigma$, where homological $(m-2)$-cycles $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)}\subset\mathbb{R}^m$ are the same as taken above. Thereby, making use of the classical method of variation of constants as in [10, 9, 8], one gets easily from (2.11) that for any $(\varphi(\eta),\psi(\xi))\in\mathcal{H}_0^*\times\mathcal{H}_0$, $(\eta,\xi)\in\Sigma\times\Sigma$,

$$\tilde{\psi}(\xi) = \mathbf{\Omega}_+\psi(\xi), \qquad \tilde{\varphi}(\eta) = \mathbf{\Omega}_+^{\circledast}\varphi(\eta), \tag{2.12}$$

where integral expressions

$$\begin{aligned}\mathbf{\Omega}_+ &:= \mathbf{1} - \int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\tilde{\psi}(\xi)\Omega_{x_0}^{-1}(\xi,\eta)\int_{\mathcal{S}_+(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}Z^{(m-1)}[\varphi(\eta),\cdot],\\ \mathbf{\Omega}_+^{\circledast} &:= \mathbf{1} - \int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\tilde{\varphi}(\eta)\Omega_{x_0}^{\circledast,-1}(\xi,\eta)\int_{\mathcal{S}_+(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\bar{Z}^{(m-1),\intercal}[\cdot,\psi(\xi)]\end{aligned} \tag{2.13}$$

are bounded Delsarte transmutation operators of Volterra type defined, correspondingly, on the whole spaces $\mathcal{H}$ and $\mathcal{H}^*$.

Now, based on operator expressions (2.13) and the definition (1.3), one gets easily the expressions for Delsarte transformed operators $\tilde{L}$ and $\tilde{L}^*$:

$$\begin{aligned}\tilde{L} &= \mathbf{\Omega}_+L\mathbf{\Omega}_+^{-1} = L + [\mathbf{\Omega}_+,L]\mathbf{\Omega}_+^{-1},\\ \tilde{L}^* &= \mathbf{\Omega}_+^{\circledast}L\mathbf{\Omega}_+^{\circledast,-1} = L^* + [\mathbf{\Omega}_+^{\circledast},L^*]\mathbf{\Omega}_+^{\circledast,-1}.\end{aligned} \tag{2.14}$$

Note also here that the transformations like (2.12) were for one-dimensional case in detail studied in [4, 17, 5]. They satisfy evidently the following easily found

conditions:

$$\tilde{L}\tilde{\psi}=0, \qquad \tilde{L}^{*}\tilde{\varphi}=0 \tag{2.15}$$

for any pair $(\tilde{\varphi},\tilde{\psi})\in\tilde{\mathcal{H}}_0^{*}\times\tilde{\mathcal{H}}_0$, which can be specified by constraints

$$\tilde{\psi}|_{\tilde{\Gamma}}=0, \qquad \tilde{\varphi}|_{\tilde{\Gamma}^{*}}=0 \tag{2.16}$$

for some hypersurface $\tilde{\Gamma}\subset\mathbb{R}^m$, related with the previously chosen hypersurface $\Gamma \subset\mathbb{R}^m$ and the homological pair of $(m-2)$-dimensional cycles $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \subset\mathbb{R}^m$. Thereby, the closed subspaces $\tilde{\mathcal{H}}_0$ and $\tilde{\mathcal{H}}_0^{*}$ can be re-defined similarly to (2.6) :

$$\begin{aligned}\tilde{\mathcal{H}}_0 &:=\{\tilde{\psi}(\xi)\in\mathcal{H}_{-}:\tilde{L}\psi(\xi)=0, \quad \tilde{\psi}(\xi)|_{\tilde{\Gamma}}=0,\ \xi\in\Sigma\subset\mathbb{C}^p\},\\ \tilde{\mathcal{H}}_0^{*} &:=\{\tilde{\varphi}(\eta)\in\mathcal{H}_{-}^{*}:\tilde{L}^{*}\tilde{\varphi}(\eta)=0, \quad \tilde{\varphi}(\eta)|_{\tilde{\Gamma}}=0,\ \eta\in\Sigma\subset\mathbb{C}^p\}\end{aligned} \tag{2.17}$$

Moreover, the following lemma, based on a pseudo-differential operators technique from [1, 17, 9] , holds.

Lemma 2.1. *The Delsarte transformed operators (2.13) by means of transmutation operators (2.12) are differential too if the starting operator $L:\mathcal{H}\longrightarrow\mathcal{H}$ was taken differential.*

As a simple consequence of the structure of the constructed above Delsarte transformed operators (2.13) one states that for any pair $(\tilde{\varphi},\tilde{\psi})\in\tilde{\mathcal{H}}_0^{*}\times\tilde{\mathcal{H}}_0$. The following differential forms equality holds:

$$\tilde{Z}^{(m-1)}[\tilde{\varphi},\tilde{\psi}]=d\tilde{\Omega}^{(m-2)}[\tilde{\varphi},\tilde{\psi}], \tag{2.18}$$

where, by definition, a pair $(\tilde{\varphi},\tilde{\psi})\in\tilde{\mathcal{H}}_0^{*}\times\tilde{\mathcal{H}}_0$ is fixed and the equality

$$\Big(<\tilde{L}^{*}\tilde{\varphi},\tilde{\psi}>-<\tilde{\varphi},\tilde{L}\tilde{\psi}>\Big)\,dx=d\tilde{Z}^{(m-1)}[\tilde{\varphi},\tilde{\psi}] \tag{2.19}$$

holds. The equality (2.17) makes it possible to construct the corresponding kernels

$$\begin{aligned}\tilde{\Omega}_x(\eta,\xi)&:=\textstyle\int_{\sigma_x^{(m-2)}}\tilde{\Omega}^{(m-2)}[\tilde{\varphi}(\eta),\tilde{\psi}(\xi)],\\ \tilde{\Omega}_{x_0}(\eta,\xi)]&:=\textstyle\int_{\sigma_{x_0}^{(m-2)}}\tilde{\Omega}^{(m-2)}[\tilde{\varphi}(\eta),\tilde{\psi}(\xi)]\end{aligned} \tag{2.20}$$

of bounded integral invertible Hilbert-Schmidt operators $\tilde{\Omega}_x,\tilde{\Omega}_{x_0}:H\to H$, and corresponding kernels

$$\begin{aligned}\tilde{\Omega}_x^{\circledast}(\eta,\xi)&:=\textstyle\int_{\sigma_x^{(m-2)}}\bar{\tilde{\Omega}}^{(m-2),\intercal}[\tilde{\varphi}(\eta),\tilde{\psi}(\xi)],\\ \tilde{\Omega}_{x_0}^{\circledast}(\eta,\xi)&:=\textstyle\int_{\sigma_{x_0}^{(m-2)}}\bar{\tilde{\Omega}}^{(m-2,\intercal)}[\tilde{\varphi}(\eta),\tilde{\psi}(\xi)]\end{aligned} \tag{2.21}$$

of bounded integral invertible Hilbert-Schmidt operators $\tilde{\Omega}_x,\tilde{\Omega}_{x_0}:H^{*}\to H^{*}$. Then the following equalities hold for all mutually related pairs $(\varphi,\psi)\in\mathcal{H}_0^{*}\times\mathcal{H}_0$ and $(\tilde{\varphi},\tilde{\psi})\in\tilde{\mathcal{H}}_0^{*}\times\tilde{\mathcal{H}}_0$:

$$\begin{aligned}\psi(\xi)&=\tilde{\psi}(\xi)\cdot\tilde{\Omega}_x^{-1}\tilde{\Omega}_{x_0},\\ \varphi(\eta)&=\tilde{\varphi}(\eta)\cdot\tilde{\Omega}_x^{\circledast,-1}\tilde{\Omega}_{x_0}^{\circledast}\end{aligned} \tag{2.22}$$

where $(\eta,\xi)\in\Sigma\times\Sigma$. Thus, based on the symmetry property between relations (2.10) and (2.21), one easily finds from expression (2.22), that expressions

$$\begin{aligned}\mathbf{\Omega}_{+}^{-1}&:=\mathbf{1}-\textstyle\int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\psi(\xi)\tilde{\Omega}_{x_0}^{-1}(\xi,\eta)\int_{\mathcal{S}_{+}(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\tilde{Z}^{(m-1)}[\tilde{\varphi}(\eta),\cdot],\\ \mathbf{\Omega}_{+}^{\circledast,-1}&:=\mathbf{1}-\textstyle\int_\Sigma d\rho(\xi)\int_\Sigma d\rho(\eta)\varphi(\eta)\tilde{\Omega}_{x_0}^{\circledast,-1}(\xi,\eta)\int_{\mathcal{S}_{+}(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\bar{\tilde{Z}}^{(m-1),\intercal}[\cdot,\tilde{\psi}(\xi)]\end{aligned} \tag{2.23}$$

for some homological $(m-2)$-dimensional cycles $\tilde{\sigma}_x^{(m-2)}, \tilde{\sigma}_{x_0}^{(m-2)} \subset \mathbb{R}^m$ are inverse to (2.14) Delsarte transmutation integral operators of Volterra type, satisfying the following relationships:

$$\psi(\xi) = \mathbf{\Omega}_+^{-1} \cdot \tilde{\psi}(\xi), \qquad \varphi(\eta) = \mathbf{\Omega}_+^{*,-1} \cdot \tilde{\varphi}(\eta) \tag{2.24}$$

for all arbitrary but fixed pairs of functions $(\varphi(\eta), \psi(\xi)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\eta), \tilde{\psi}(\xi)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $(\eta, \xi) \in \Sigma \times \Sigma$. Thus, one can formulate the following characterizing the constructed Delsarte transmutation operators theorem.

Theorem 2.2. *Let a matrix multi-dimensional differential operator (2.1) acting in a Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m; \mathbb{C}^N)$ and its formally adjoint operator (2.2) acting in a Hilbert space $\mathcal{H}^* = L_2^*(\mathbb{R}^m; \mathbb{C}^N)$, possess, correspondingly, a pair of closed spaces $\mathcal{H}_0$ and $\mathcal{H}_0^*$ (2.6) of their generalized kernel eigenfunctions parametrized by some set $\Sigma \subset \mathbf{C}^p$. Then there exist bounded invertible Delsarte transmutation integral operators $\mathbf{\Omega}_+ : \mathcal{H} \longrightarrow \mathcal{H}$ and $\Omega_+^{\circledast} : \mathcal{H}^* \longrightarrow \mathcal{H}^*$, such that for this pair $(\mathcal{H}_0, \mathcal{H}_0^*)$ (2.6) of closed subspaces (2.6) and their dual ones (2.17) the corresponding bounded invertible mappings (2.13) $\mathbf{\Omega}_+ : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0$ and $\mathbf{\Omega}_+^{\circledast} : \mathcal{H}_0^* \to \tilde{\mathcal{H}}_0^*$ are compatibly defined. Moreover, the operator expressions (2.14) are also differential, acting in the corresponding spaces $\mathcal{H}$ and $\mathcal{H}^*$.*

The revealed above structure of the Delsarte transmutation operators (2.13) makes it possible to understand more deeply their properties by means of deriving new integral equations being multi-dimensional analogs of the well known Gelfand-Levitan-Marchenko equations [4, 5, 17, 6], that will be a topic of the next chapter.

3. Multi-dimensional Gelfand-Levitan-Marchenko type integral equations

Investigating the inverse scattering problem for a three-dimensional perturbed Laplace operator

$$L(x; \partial) = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} + q(x), \tag{3.1}$$

with $q \in W_2^2(\mathbb{R}^3)$, $x \in \mathbb{R}^3$, in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C})$, L.D. Faddeev in [15] has suggested some approach to studying the structure of the corresponding Delsarte transmutation operators $\mathbf{\Omega}_\gamma : \mathcal{H} \to \mathcal{H}$ of Volterra type, based on a priori chosen hypersurfaces $S_{\pm\gamma}^{(x)} = \{y \in \mathbb{R}^3 : < y - x, \pm\gamma > > 0\}$, parametrized by unity vectors $\gamma \in \mathbb{S}^2$, where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the standard two-dimensional sphere imbedded into $\mathbb{R}^3$. Making use of these Delsarte transmutation operators of Volterra type, in [15] there was derived some three-dimensional analog of the integral GLM-equation, whose solution gives rise to the kernel of the corresponding Delsarte transmutation operator for (3.1) . But the important two problems related with this approach were not discussed in detail: the first one concerns the question whether the Delsarte transformed operator $\tilde{L} = \mathbf{\Omega}_\gamma L \hat{\mathbf{\Omega}}_\gamma^{-1}$ is also a differential operator of Laplace type, and the second one concerns the question of existing Delsarte transmutation operators in the Faddeev form.

Below we will study our multi-dimensional Delsarte transmutation operators (2.14), parametrized by a hypersurface $S_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})$ piecewise smoothly imbedded into $\mathbb{R}^m$.

Consider now some $(m-2)$-dimensional homological cycles $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \subset \mathbb{R}^m$ and two $(m-1)$-dimensional smooth hypersurfaces

$$S_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}), \quad S_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})$$

spanned between them in such a way that the whole hypersurface $\mathcal{S}_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \cup \mathcal{S}_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})$ is closed. Then similarly to the construction of Chapter 3 one can naturally define two pairs of Delsarte transmutation operators for a given pair of multi-dimensional differential operators (2.1) and (2.9), namely operators $\mathbf{\Omega}_+ : \mathcal{H} \rightleftharpoons \mathcal{H}$, $\mathbf{\Omega}_+^{\circledast} : \mathcal{H}^* \rightleftharpoons \mathcal{H}^*$, defined by (2.13), and operators $\mathbf{\Omega}_-^{\circledast} : \mathcal{H}^* \rightleftharpoons \mathcal{H}^*$, $\mathbf{\Omega}_-^{\circledast} : \mathcal{H}^* \rightleftharpoons \mathcal{H}^*$, where, by definition,

(3.2)

$$\begin{aligned} \mathbf{\Omega}_- &:= \mathbf{1} - \int_\Sigma d\rho(\xi) \int_\Sigma d\rho(\eta) \tilde{\psi}(\xi) \Omega_{x_0}^{-1}(\xi,\eta) \int_{\mathcal{S}_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\eta), \cdot], \\ \mathbf{\Omega}_-^{\circledast} &:= \mathbf{1} - \int_\Sigma d\rho(\xi) \int_\Sigma d\rho(\eta) \tilde{\varphi}(\eta) \Omega_{x_0}^{\circledast,-1}(\xi,\eta) \int_{\mathcal{S}_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[\cdot, \psi(\xi)] \end{aligned}$$

Subject to the Delsarte transmutation operators (2.12)) and (3.2) the following operator relationships

$$\tilde{L} = \mathbf{\Omega}_\pm L \mathbf{\Omega}_\pm^{-1}, \quad \mathbf{\Omega}_\pm^{\circledast} L_\pm^{\circledast} \mathbf{\Omega}_\pm^{\circledast,-1} = \tilde{L}^*$$

hold. As in theory of classical GLM-equations [4, 5, 17], we can now construct linear integral operators of Fredholm type $\mathbf{\Phi} : \mathcal{H} \to \mathcal{H}$, $\mathbf{\Phi}^{\circledast} : \mathcal{H}^* \to \mathcal{H}^*$ of Fredholm type, such that

$$\mathbf{1} + \mathbf{\Phi} := \mathbf{\Omega}_+^{-1} \cdot \mathbf{\Omega}_-, \qquad \mathbf{1} + \mathbf{\Phi}^{\circledast} := \mathbf{\Omega}_+^{\circledast,-1} \cdot \mathbf{\Omega}_-^{\circledast}. \tag{3.3}$$

Making use of the expressions (3.3), one easily gets a pair of linear integral GLM-equations:

$$\mathbf{\Omega}_+ \cdot (\mathbf{1} + \mathbf{\Phi}) = \mathbf{\Omega}_-, \qquad \mathbf{\Omega}_+^{\circledast} \cdot (\mathbf{1} + \mathbf{\Phi}^{\circledast}) = \mathbf{\Omega}_-^{\circledast}, \tag{3.4}$$

whose solution is a pair of the corresponding Volterra type kernels for the Delsarte transmutation operators $\mathbf{\Omega}_+$ and $\mathbf{\Omega}_+^{\circledast}$. Thus, the problem of constructing Delsarte transmutation operators for a given pair of differential operators (2.1) and (2.9) is reduced to that of describing a suitable class of linear Fredholm type operators (3.3) in the Hilbert space $\mathcal{H}$, satisfying the following natural conditions: operators $(\mathbf{1} + \mathbf{\Phi}) : \mathcal{H} \longrightarrow \mathcal{H}$ and $(\mathbf{1} + \mathbf{\Phi}^{\circledast}) : \mathcal{H}^* \longrightarrow \mathcal{H}^*$ are onto, bounded and invertible and, moreover,

$$(\mathbf{1} + \mathbf{\Phi})L = L(\mathbf{1} + \mathbf{\Phi}), \quad (\mathbf{1} + \mathbf{\Phi}^{\circledast})L^* = L^*(\mathbf{1} + \mathbf{\Phi}^{\circledast}) \tag{3.5}$$

due to (3.4) and (2.13) . This problem is very important for the theory devised here could be effectively applied to studying diverse spectral properties of a given pair of Delsarte transformed differential operators (2.1) and (2.9) and is planned to be studied in detail in another place.

4. The structure of Delsarte transmutation operators for affine pencils of multidimensional differential expressions

Consider in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m; \mathbb{C}^N)$ an affine polynomial in $\lambda \in \mathbb{C}$ pencil of multi-dimensional differential operators

$$L(x; \partial|\lambda) := \sum_{i=0}^{r(L)} \lambda^i L_i(x; \partial), \tag{4.1}$$

where $x \in \mathbb{R}^m$, $ordL_i(x;\partial) = n_i \in \mathbb{Z}_+$, $i = \overline{1, r(L)}$, the order $r(L) \in \mathbb{Z}_+$ is fixed and

$$L_i(x;\partial) := \sum_{|\alpha_i|=0}^{n_i} a_{i,\alpha_i}(x) \frac{\partial^{|\alpha_i|}}{\partial x^{\alpha_i}} \tag{4.2}$$

are differential expressions with smooth coefficients $a_{i,\alpha_i} \in \mathcal{S}(\mathbb{R}^m; End\mathbb{C}^N)$, $i = \overline{1, r(L)}$. The pencil (4.1) can be, in particular, characterized by its spectrum

$$\sigma(L) = \{\lambda \in \mathbb{C} : \exists\ \psi(x;\lambda) \in \mathcal{H}_-,\quad L(x;\partial|\lambda)\psi(x;\lambda) = 0\}. \tag{4.3}$$

As was demonstrated in [7]. the transformations of pencil (4.1) which preserve a part of the spectrum $\sigma(L)$ and simultaneously change in a prescribed way the rest of the spectrum (so called an assignment spectrum problem [7]) are of very importance for feedback control theory and its applications in different fields of mechanics.

We will try here to interpret these "spectrum assignment" transformations as ones of Delsarte transmutation type, satisfying some additional special conditions. Thus, we look for such a transformation $\mathbf{\Omega} : \mathcal{H} \longrightarrow \mathcal{H}$ of the pencil (4.1) into a similar pencil

$$\tilde{L}(x;\partial|\lambda) = \sum_{i=1}^{r(L)} \lambda^i \tilde{L}_i(x;\partial), \qquad \tilde{L}_i(x;\partial) := \sum_{|\alpha_i|=0}^{n_i} \tilde{a}_{i,\alpha_i}(x) \frac{\partial^{|\alpha_i|}}{\partial x^{\alpha_i}} \tag{4.4}$$

with $\tilde{a}_{i,\alpha_i} \in S(\mathbb{R}^m; End\mathbb{C}^N)$, $i = \overline{1, r(L)}$, $\lambda \in \mathbb{C}$, of the same polynomial and differential orders, that

$$\tilde{L} = L + [\mathbf{\Omega}, L]\mathbf{\Omega}^{-1} = \mathbf{\Omega} L \mathbf{\Omega}^{-1}. \tag{4.5}$$

For such an operator $\mathbf{\Omega} : \mathcal{H} \to \mathcal{H}$ to be constructed, we suggest to extend the polynomial pencil of differential operators (4.1) to a pure differential operator $L_\tau := L(x;\partial|\partial/\partial\tau)$, $x \in \mathbb{R}^m$, $\tau \in \mathbb{R}$, with $\mathbb{R} \ni \tau$-independent coefficients and acting suitably in the parametric functional space $\mathcal{H}_{(\tau)} := L_1(\mathbb{R}_\tau; \mathcal{H})$. Thereby we have come at the same situation, which was studied before in [11]. For completeness, we shall deliver here a short derivation of the corresponding affine expression for the Delsarte transmutation operator $\mathbf{\Omega} : \mathcal{H} \longrightarrow \mathcal{H}$.

Let a pair of functions $(\varphi_{(\tau)}, \psi_{(\tau)}) \in \mathcal{H}^*_{(\tau)} \times \mathcal{H}_{(\tau)}$ be arbitrary and consider the following semi-linear scalar form on $\mathcal{H}^*_{(\tau)} \times \mathcal{H}_{(\tau)}$:

$$(\varphi_{(\tau)}, \psi_{(\tau)}) := \int_{\mathbb{R}_\tau} d\tau \int_{\mathbb{R}^m} dx \bar{\varphi}^\intercal_{(\tau)}(x) \psi_{(\tau)}(x). \tag{4.6}$$

Then subject to the internal semi-linear form $< \cdot, \cdot >$ on $\mathbb{C}^N \times \mathbb{C}^N$ one can write down for the operator $L_{(\tau)} : \mathcal{H}_{(\tau)} \longrightarrow \mathcal{H}_{(\tau)}$ and any pair $(\varphi_{(\tau)}, \psi_{(\tau)}) \in \mathcal{H}^*_{(\tau)} \times \mathcal{H}_{(\tau)}$ the following Lagrangian identity:

$$\Big[< L^*_{(\tau)}\varphi_{(\tau)}, \psi_{(\tau)} > - < \varphi_{(\tau)}, L_{(\tau)}\psi_{(\tau)} > \Big] d\tau \wedge dx = dZ^{(m)}_{(\tau)}[\varphi, \psi], \tag{4.7}$$

where $Z^{(m)}_{(\tau)}[\varphi, \psi] \in \Lambda^m(\mathbb{R}_\tau \times \mathbb{R}^m; \mathbb{C})$ is the corresponding differential m-form with values in $\mathbb{C}$, parametrically depending on $\tau \in \mathbb{R}$. Thus, for defining the closed

subspaces $\mathcal{H}^*_{(\tau),0} \subset \mathcal{H}^*_{(\tau),-}$ one can write down, correspondingly, the following expressions:

$$\begin{aligned} &\mathcal{H}_{(\tau),0} := \{\psi_{(\tau)}(\xi) \in \mathcal{H}_{(\tau),-} : L_{(\tau)}\psi_{(\tau)}(\xi) = 0, \\ &\quad \tau \in \mathbb{R},\ \psi_{(\tau)}(\xi)|_{\Gamma} = 0,\ \xi \in \Sigma \subset \mathbb{C}^p\}, \\ &\mathcal{H}^*_{(\tau),0} := \{\varphi_{(\tau)}(\eta)(\in \mathcal{H}^*_{(\tau),-} : L^*_{(\tau)}\varphi_{(\tau)}(\eta) = 0, \\ &\quad \tau \in \mathbb{R},\ \ \varphi_{(\tau)}(\eta)|_{\Gamma} = 0,\ \eta \in \Sigma \subset \mathbb{C}^p\}, \end{aligned} \tag{4.8}$$

where $\Gamma \subset\mathbb{R}^m$ is some piecewise smooth boundary hypersurfaces in $\mathbb{R}^m$. Similar expressions one can also write down for the Delsarte transformed operator expression $\tilde{L} : \mathcal{H}_{(\tau)} \longrightarrow \mathcal{H}_{(\tau)}$:

$$\begin{aligned} &\tilde{\mathcal{H}}_{(\tau),0} := \{\tilde{\psi}_{(\tau)}(\xi) \in \mathcal{H}_{(\tau),-} : \tilde{L}_{(\tau)}\tilde{\psi}_{(\tau)}(\xi) = 0, \\ &\quad \tau \in \mathbb{R},\ \tilde{\psi}_{(\tau)}(\xi)|_{\tilde{\Gamma}} = 0,\ \xi \in \Sigma \subset \mathbb{C}^p\}, \\ &\tilde{\mathcal{H}}^*_{(\tau),0} := \{\tilde{\varphi}_{(\tau)}(\eta) \in \mathcal{H}^*_{(\tau),-} : \tilde{L}^*_{(\tau)}\tilde{\varphi}_{(\tau)}(\eta) = 0, \\ &\quad \tau \in \mathbb{R},\ \ \tilde{\varphi}_{(\tau)}(\eta)|_{\tilde{\Gamma}} = 0,\ \eta \in \Sigma \subset \mathbb{C}^p\}, \end{aligned} \tag{4.9}$$

where $\tilde{\Gamma} \subset\mathbb{R}^3$ is some piecewise smooth boundary hypersurface in $\mathbb{R}^m$.

Making use of the expressions (4.6) and (4.7), we easily find that the differential m-form $Z^{(m)}_{(\tau)}[\varphi,\psi] \in \Lambda^m(\mathbb{R}_\tau \times \mathbb{R}^m;\mathbb{C})$ is exact for any pair $(\varphi,\psi) \in \mathcal{H}^*_{(\tau),0} \times \mathcal{H}_{(\tau),0}$. This means due to the Poincare lemma [12, 13], that there exists a differential $(m-1)$-form $\Omega^{(m-1)}_\tau[\varphi,\psi] \in \Lambda^{m-1}(\mathbb{R}_\tau \times \mathbb{R}^m;\mathbb{C})$, such that

$$Z^{(m)}_{(\tau)}[\varphi,\psi] = d\Omega^{(m-1)}_\tau[\varphi,\psi] \tag{4.10}$$

for all pairs $(\varphi_{(\tau)},\psi_{(\tau)}) \in \mathcal{H}^*_{(\tau),0} \times \mathcal{H}_{(\tau),0}$. Now we are in a starting position for defining the corresponding Delsarte transmutation operators $\mathbf{\Omega}_{(\tau)} : \mathcal{H}_{(\tau),0} \to \tilde{\mathcal{H}}_{(\tau),0}$ and $\mathbf{\Omega}^{\circledast}_{(\tau)} : \mathcal{H}^*_{(\tau),0} \to \tilde{\mathcal{H}}^*_{(\tau),0}$:

$$\begin{aligned} &\tilde{\psi}_{(\tau)}(\xi) = \mathbf{\Omega}_{(\tau)} \cdot \psi_{(\tau)}(\xi) := \psi_{(\tau)}(\xi) \cdot \Omega^{-1}_{(x,\tau)}\Omega_{(x_0,\tau)} = \\ &(\mathbf{1} - \tilde{\psi}_{(\tau)}\Omega^{-1}_{(x_0,\tau)} \textstyle\int_{\mathcal{S}(\sigma^{(m-1)}_x,\sigma^{(m-1)}_{x_0})} Z^{(m)}_{(\tau)}[\varphi_{(\tau)},\cdot])\psi_{(\tau)}(\xi), \\ &\tilde{\varphi}_{(\tau)}(\eta) = \mathbf{\Omega}^{\circledast}_{(\tau)} \cdot \varphi_{(\tau)}(\eta) := \varphi_{(\tau)}(\eta) \cdot \Omega^{\circledast,-1}_{(x,\tau)}\Omega^{\circledast}_{(x_0,\tau)} = \\ &(\mathbf{1} - \tilde{\varphi}_{(\tau)}\Omega^{\circledast,-1}_{(x_0,\tau)} \textstyle\int_{\mathcal{S}(\sigma,^{(m-1)}_x,\sigma^{(m-1)}_{x_0})} \bar{Z}^{(m),\intercal}_{(\tau)}[\cdot,\psi_{(\tau)}])\varphi_{(\tau)}(\eta), \end{aligned} \tag{4.11}$$

Here $(\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)) \in \mathcal{H}^*_{(\tau),0} \times \mathcal{H}_{(\tau),0},\ \xi,\eta \in \Sigma$, and, due to (4.9), for kernels $\Omega_{(x,\tau)}(\eta,\xi),\ \Omega_{(x_0,\tau)}(\eta,\xi) \in H \otimes H$ and $\Omega^{\circledast}_{(x,\tau)}(\eta,\xi),\ \Omega^{\circledast}_{(x_0,\tau)}(\eta,\xi) \in H^* \otimes H^*$ of the corresponding integral operators $\Omega_{(x,\tau)},\ \Omega_{(x_0,\tau)} : H \to H$ and $\Omega^{\circledast}_{(x,\tau)},\ \Omega^{\circledast}_{(x_0,\tau)} : H^* \to H^*$ one has

$$\begin{aligned} &\textstyle\int_{\mathcal{S}(\sigma^{(m-1)}_x,\sigma^{(m-1)}_{x_0})} Z^{(m)}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] = \int_{\sigma^{(m-1)}_x} \Omega^{(m-1)}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] \\ &- \textstyle\int_{\sigma^{(m-1)}_{x_0}} \Omega^{(m-1)}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] := \Omega_{(x,\tau)}(\eta,\xi) - \Omega_{(x_0,\tau)}(\eta,\xi), \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\textstyle\int_{\mathcal{S}(\sigma,^{(m-1)}_x,\sigma^{(m-1)}_{x_0})} \bar{Z}^{(m),\intercal}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] = \int_{\sigma^{(m-1)}_x} \bar{\Omega}^{(m-1),\intercal}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] \\ &- \textstyle\int_{\sigma^{(m-1)}_{x_0}} \bar{\Omega}^{(m-1),\intercal}_{(\tau)}[\varphi_{(\tau)}(\eta),\psi_{(\tau)}(\xi)] := \Omega^{\circledast}_{(x,\tau)}(\eta,\xi) - \Omega^{\circledast}_{(x_0,\tau)}(\eta,\xi), \end{aligned}$$

where, as before, $S(\sigma^{(m-1)}_x,\sigma^{(m-1)}_{x_0}) \subset \mathbb{R}^m$ is a smooth hypersurface in the configuration space $\mathbb{R}^m$, spanned between two arbitrary but fixed nonintersecting each other homological $(m-1)$-dimensional cycles $\sigma^{(m-1)}_x$and $\sigma^{(m-1)}_{x_0} \subset \mathbb{R}^m$, parametrized

by points $x,\ x_0 \in \mathbb{R}^m$. As a result of the construction above, the Volterra type integral operators

(4.13)
$$\mathbf{\Omega}_{(\tau)} := \mathbf{1} - \int_\Sigma d\rho(\xi) \int_\Sigma d\rho(\eta) \tilde{\psi}_{(\tau)}(\xi) \Omega^{-1}_{(x_0,\tau)}(\xi,\eta) \int_{\mathcal{S}_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1)}_{(\tau)}[\varphi_{(\tau)}(\eta), \cdot],$$

and

(4.14)
$$\mathbf{\Omega}^{\circledast}_{(\tau)} := \mathbf{1} - \int_\Sigma d\rho(\xi) \int_\Sigma d\rho(\eta) \tilde{\varphi}_{(\tau)}(\eta) \Omega^{\circledast,-1}_{(x_0,\tau)}(\xi,\eta) \int_{\mathcal{S}_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{()m-2)})} \bar{Z}^{(m-1),\intercal}_{(\tau)}[\cdot, \psi_{(\tau)}(\xi)]$$

being bounded and invertible act, correspondingly, in the spaces $\mathcal{H}_{(\tau)}$ and $\mathcal{H}^*_{(\tau)}$. Moreover, the Delsarte transformed operator $\tilde{L}_{(\tau)} : \mathcal{H}_{(\tau)} \longrightarrow \mathcal{H}_{(\tau)}$ can be written down as

$$\tilde{L}_{(\tau)} = \mathbf{\Omega}_{(\tau)} L_{(\tau)} \mathbf{\Omega}^{-1}_{(\tau)} = L_{(\tau)} + [\mathbf{\Omega}_{(\tau)}, L_{(\tau)}]\mathbf{\Omega}^{-1}_{(\tau)}, \tag{4.15}$$

being, due to reasoning as in [9, 8], also a differential multi-dimensional operator in $\mathcal{H}_{(\tau)}$.

Now we can make the drawback reduction of our τ-dependent objects, recalling, that our operator (4.1) doesn't depend on the parameter $\tau \in \mathbb{R}$. In particular, from (4.8) one can get that for any $(\varphi_{(\tau)}(\eta), \psi_{(\tau)}(\xi)) \in \mathcal{H}^*_{(\tau),0} \times \mathcal{H}_{(\tau),0}$, $\xi, \eta \in \Sigma$,

$$\psi_{(\tau)}(\xi) = \psi_\lambda(\xi) e^{\lambda\tau}, \qquad \varphi_{(\tau)}(\eta) = \varphi_\lambda(\eta) e^{-\bar{\lambda}\tau} \tag{4.16}$$

with $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$ and any pair $(\varphi_\lambda(\xi), \psi_\lambda(\xi)) \in \mathcal{H}^*_0 \times \mathcal{H}_0$, $\xi, \eta \in \Sigma_\sigma$,

$$\begin{gathered} \mathcal{H}_0 := \{\psi_\lambda(\xi) \in \mathcal{H}_- : L(x;\partial|\lambda)\psi_\lambda(\xi) = 0, \\ \psi_\lambda(\xi)|_\Gamma = 0,\ (\lambda;\xi) \in \sigma(L) \cap \bar{\sigma}(L^*) \times \Sigma_\sigma\}, \\ \mathcal{H}^*_0 := \{\varphi_\lambda(\eta) \in \mathcal{H}^*_- : L^*(x;\partial|\lambda)\varphi_\lambda(\eta) = 0, \\ \varphi_\lambda(\eta)|_\Gamma = 0,\ (\lambda;\eta) \in \sigma(L) \cap \bar{\sigma}(L^*) \times \Sigma_\sigma\}, \end{gathered} \tag{4.17}$$

where, by definition, $\Sigma_\sigma \times \mathbb{C} \subset \Sigma$ is some "spectral"set of parameters. With respect to the closed subspaces $\mathcal{H}_0 \subset \mathcal{H}_-$ and $\mathcal{H}^*_0 \subset \mathcal{H}^*_-$ the corresponding Delsarte transmutation operators $\mathbf{\Omega} : \mathcal{H} \to \mathcal{H}$ and $\mathbf{\Omega}^{\circledast} : \mathcal{H}^* \to \mathcal{H}^*$ can be retrieved easily, making use of the expressions (4.15), substituted into (4.12) and (4.13) :

$$\begin{gathered} \mathbf{\Omega} := \mathbf{1} - \int_{\sigma(L)\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\eta) \tilde{\psi}_\lambda(\xi) \\ \times \Omega^{-1}_{x_0}(\lambda;\xi,\eta) \int_{\mathcal{S}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)})} Z^{(m)}[\varphi_\lambda(\eta), \cdot], \\ \mathbf{\Omega}^{\circledast} := \mathbf{1} - \int_{\sigma(L)\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) \int_{\sigma(L)\cap\bar{\sigma}(L^*)} \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\eta) \tilde{\varphi}_\lambda(\eta) \\ \times \Omega^{\circledast,-1}_{x_0}(\lambda;\xi,\eta) \int_{\mathcal{S}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)})} Z^{(m)}[\cdot, \psi_\lambda(\xi)], \end{gathered} \tag{4.18}$$

where $d\rho_\sigma \times d\rho_{\Sigma_\sigma}$ is the corresponding finite Borel measure on Borel subsets of $\sigma(L) \cap \bar{\sigma}(L^*) \times \Sigma_\sigma \subset \Sigma$,

$$\begin{gathered} \tilde{\psi}_\lambda(\xi) := \psi_\lambda(\xi) \cdot \Omega_x^{-1} \Omega_{x_0} \\ \tilde{\varphi}_\lambda(\eta) := \varphi_\lambda(\eta) \cdot \Omega_x^{\circledast,-1} \Omega^{\circledast}_{x_0}, \end{gathered} \tag{4.19}$$

and due to semi-linearity, the expressions for kernels

$$\begin{gathered} \Omega_x(\lambda;\xi,\eta) := \Omega_{(x,\tau)}[\varphi_\lambda e^{-\bar{\lambda}\tau}, \psi_\lambda e^{\lambda\tau}], \\ \Omega_{x_0}[\varphi_\lambda, \psi_\lambda] := \Omega_{(x_0,\tau)}[\varphi_\lambda e e^{-\bar{\lambda}\tau}, \psi_\lambda e^{\lambda\tau}], \\ Z^{(m)}[\varphi_\lambda, \psi_\lambda] := Z^{(m)}_{(\tau)}[\varphi_\lambda e^{-\bar{\lambda}\tau}, \psi_\lambda e^{\lambda\tau}], \end{gathered} \tag{4.20}$$

don't depend on the whole on the parameter $\tau \in \mathbb{R}_\tau$ but only on $(\lambda;\xi) \in \sigma(L) \cap \bar{\sigma}(L^*) \times \Sigma_\sigma$. Moreover, if one to write down the differential m-form $Z^{(m)}[\varphi_\lambda, \psi_\lambda] \in \Lambda^m(\mathbb{R}_\tau \times \mathbb{R}^m;\mathbb{C})$, $(\varphi_\lambda(\xi), \psi_\lambda(\xi)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\xi, \eta \in \Sigma_\sigma$, as

$$\begin{aligned} Z^{(m)}[\varphi_\lambda, \psi_\lambda] &= \sum_{i=1}^{m} dx_1 \wedge dx_2 \wedge ... \wedge dx_{i-1} \wedge Z_i[\varphi_\lambda, \psi_\lambda] d\tau \wedge dx_{i+1} \wedge \\ & ... \wedge dx_m + Z_0[\varphi_\lambda, \psi_\lambda] dx, \end{aligned} \tag{4.21}$$

then, due to the specially chosen $(m-1)$-dimensional homological cycles $(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)})$ and the corresponding closed m-dimensional surface $S(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)}) = \mathbb{R}^m$ at which $d\tau = 0$, the differential m-forms $Z^{(m)}[\varphi_\lambda, \psi_\lambda] \in \Lambda^m(\mathbb{R}^m;\mathbb{C})$, $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$, bring about the following expressions:

$$Z^{(m)}[\varphi_\lambda, \psi_\lambda] = Z_0[\varphi_\lambda, \psi_\lambda] dx, \tag{4.22}$$

where, by definition, for any $i = \overline{0, m}$

$$Z_i[\varphi_\lambda, \psi_\lambda] := Z_{i,(\tau)}[\varphi_\lambda e^{-\bar{\lambda}\tau}, \psi_\lambda e^{\lambda\tau}], \tag{4.23}$$

being, evidently, not dependent more on the parameter $\tau \in \mathbb{R}$ but only on $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$. Thus due to (4.21) and (4.22) one can finally write down Delsarte transmutation operators (4.12) and (4.13) as the following invertible and bounded of Volterra type integral expressions:

$$\begin{aligned} \mathbf{\Omega} &:= \mathbf{1} - \int_{\sigma(L)\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\eta) \tilde{\psi}_\lambda(\xi) \\ & \times \Omega_{x_0}^{-1}(\lambda;\xi,\eta) \int_{\mathcal{S}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)})} Z_{(0)}^{(m)}[\varphi_\lambda(\eta), \cdot] dx, \\ \mathbf{\Omega}^{\circledast} &:= \mathbf{1} - \int_{\sigma(L)\cap\bar{\sigma}(L^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi) \int_{\sigma(L)\cap\bar{\sigma}(L^*)} \int_{\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\eta) \tilde{\varphi}_\lambda(\eta) \\ & \times \Omega_{x_0}^{\circledast,-1}(\lambda;\xi,\eta) \int_{\mathcal{S}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)})} Z_{(0)}^{(m)}[\cdot, \psi_\lambda(\xi)] dx, \end{aligned} \tag{4.24}$$

where, by definition, $(\varphi_\lambda, \psi_\lambda) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $(\tilde{\varphi}_\lambda, \tilde{\psi}_\lambda) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$ and $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$. The operator expressions (4.24) were defined before correspondingly, on closed subspaces of generalized eigenfunctions $\mathcal{H}_0$ and $\mathcal{H}_0^*$. In the case when these spaces are dense, correspondingly, in ambient spaces $\mathcal{H}_-$ and $\mathcal{H}_-^*$, the operator expressions (4.24) can be naturally extended, correspondingly, upon the whole Hilbert spaces $\mathcal{H}_-$ and $\mathcal{H}_-^*$ as also invertible integral operators of Volterra type and, thereby, can be defined correspondingly on $\mathcal{H}$ and $\mathcal{H}^*$ due to duality [19, 1] between Hilbert spaces $\mathcal{H}_-$ and $\mathcal{H}_+$, where the latter space is dense in $\mathcal{H}$. The same way as in Chapter 3 above one can construct the corresponding pair (3.4) of Gelfand-Levitan-Marchenko integral equations for an affine polynomial pencil (4.1) of multidimensional differential operators in the Hilbert space $\mathcal{H}$. The latter note finishes our present analysis of the structure of Delsarte transmutation operators for pencils of multidimensional differential operators. Concerning their natural applications to the inverse spectral problem and related problems of feedback control theory mentioned before, we plan to stop on them in more detail later.

5. Acknowledgements

Authors are cordially thankful to prof. B.N. Datta (USA, Illinois University), prof. D.L. Blackmore (Newark USA, NJIT), prof. Nizhnik L.P. (Kyiv, Inst. of Math.at NAS), prof. T. Winiarska (Krakow, PK), profs. A. Pelczar and J. Ombach (Krakow, UJ) and prof. Z. Peradzynski (Warszawa, UW) for valuable discussions of problems studied in the work.

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